Differential Equations I Cheat Sheet (A Level Only)

Classifying Differential Equations

Differential equations are incredibly useful and can be used to model various phenomena since they describe the rate of change of a quantity with respect to another quantity. In other words, how the change of an independent variable affects a dependent variable. Generally they are difficult to solve; however in some cases there are methods to solve differential equations of appropriate forms. As a result, differential equations can be classified using the following terms:

- Order – the greatest number of times the dependent variable has been differentiated
- Linear – the dependent variable features in the differential equation only as a power of 0 and 1
- Homogeneous – every term of the differential equation involves the dependent variable (this definition is specific to the AQA course as homogeneous can also refer to a different property of differential equations)

Example 1: Classify the following differential equations:

a) $\frac{d^4y}{dx^4} + 2x\frac{d^2y}{dx^2} + 4 = 0$

b) $y\frac{d^2y}{dx^2} + x^3y = 0$

c) $\frac{dy}{dx} + x^5y = \sin x$

a) The term with the highest derivative is $\frac{d^4y}{dx^4}$. All terms have y as a power of 0 or 1. The 4 term does not depend on y.	So the differential equation is 4th order, linear, non-homogenous
b) The term with the highest derivative is $y\frac{d^2y}{dx^2}$. In the $y\frac{d^2y}{dx^2}$ term y has a power of 2. Every term depends on y.	So the differential equation is 2nd order, non-linear, homogenous
c) The term with the highest derivative is $\frac{dy}{dx}$. All terms have y as a power of 0 or 1. The $\sin x$ term does not depend on y.	So the differential equation is 1st order, linear, non-homogenous

Integrating Factors

An integrating factor can be used to solve 1st order linear inhomogeneous differential equations of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

This method works by first multiplying both sides of the equation by the integrating factor $I(x)$:

$$I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x)$$

By choosing $I(x)$ such that $I'(x) = I(x)P(x)$, the equation above can be simplified by applying the product rule:

$$\frac{d}{dx}(I(x)y) = I(x)Q(x)$$

Integrating both sides and rearranging gives the solution:

$$y = \frac{1}{I(x)}\int I(x)\,Q(x)\,dx$$

An explicit formula for the integrating factor $I(x)$ can be found by separating the variables on the differential equation $I'(x) = I(x)P(x)$. This formula is:

$$I(x) = e^{\int P(x)\,dx}$$

General and Particular Solutions of Differential Equations

Solving a differential equation hinges on taking an equation with derivatives in and removing some of them. This requires integration which introduces arbitrary constants. The general solution is the solution with all the arbitrary constants. With an additional condition for each arbitrary constant, these constants can be fixed to give the particular solution. To solve an n^{th} order differential equation, n integrations are required so there are n arbitrary constants.

Example 2: Find the general solution to the following differential equation:

$$3x\frac{dy}{dx} + y = 4x^2$$

Classify the equation to decide what approach to take.	1st order linear inhomogeneous differential equation so use an integrating factor
Put into the correct form by dividing both sides by $3x$.	$\frac{dy}{dx} + \frac{y}{3x} = \frac{4}{3}x$
Use $I(x) = e^{\int P(x)\,dx}$ to find $I(x)$. Notice that there is no need to add a constant here as it would cancel out later.	$I(x) = e^{\int \frac{1}{3x}dx} = e^{\frac{1}{3}\ln x} = x^{\frac{1}{3}}$
Multiply both sides of the differential equation by $x^{\frac{1}{3}}$.	$x^{\frac{1}{3}}\frac{dy}{dx} + \frac{y}{3x^{\frac{2}{3}}} = \frac{4}{3}x^{\frac{4}{3}}$
Rewrite LHS using the product rule.	$\frac{d}{dx}\left(x^{\frac{1}{3}}y\right) = \frac{4}{3}x^{\frac{4}{3}}$
Integrate both sides, adding only one necessary constant of integration.	$x^{\frac{1}{3}}y = \frac{4}{7}x^{\frac{7}{3}} + c$
Divide both sides by $x^{\frac{1}{3}}$.	$y = \frac{4}{7}x^2 + cx^{-\frac{1}{3}}$

Example 3: Consider the differential equation below:

$$\frac{dx}{dt} + 2tx = e^{-t^2}$$

a) Find the general solution.

b) Find the particular solution given that $x(0) = 5$.

a) Classify the equation to decide what approach to take.	1st order linear inhomogeneous differential equation so use an integrating factor
Use $I(t) = e^{\int P(t)\,dt}$ to find $I(t)$.	$I(t) = e^{\int 2t\,dt} = e^{t^2}$
Multiply both sides of the differential equation by e^{t^2}.	$e^{t^2}\frac{dx}{dt} + 2te^{t^2}x = 1$
Rewrite LHS using the product rule.	$\frac{d}{dt}\left(e^{t^2}x\right) = 1$
Integrate both sides.	$e^{t^2}x = t + c$
Divide both sides by e^{t^2} to obtain the general solution.	$x = e^{-t^2}(t + c)$
b) Substitute in initial conditions to fix the arbitrary constant.	$(5) = e^{-(0)^2}(0 + c) = c$
Substitute $c = 5$ into general solution to get the particular solution.	$x = e^{-t^2}(t + 5)$

Differential Equation Modelling

Differential equations provide an excellent way to model the changing dynamics of the real world. The solutions to these differential equations can then provide huge insight into the situation being modelled.

Example 4: A doctor gives a patient 270mg of drug X. After 2 weeks the patient takes a blood test and finds they have 30mg of drug X still in their blood. If, for some constant k to be found, the mass of drug X is modelled by the differential equation:

$$\frac{dX}{dt} = kX$$

a) Find the particular solution for the mass of drug X after t weeks, where $t \geq 0$.

b) What is the half-life of drug X in days to 3 significant figures?
(Where half-life is the time for the mass of drug X to half)

a) Find the general solution by separating variables. Where $\ln A$ is an arbitrary constant.	$\int \frac{1}{X}dX = \int k\,dt \Rightarrow \ln X = kt + \ln A$ $\Rightarrow X = Ae^{kt}$
Substitute in known information to find A and k.	$X(0) = 270 \Rightarrow 270 = Ae^{k(0)} = A$ $X(2) = 30 \Rightarrow 30 = 270e^{2k}$ $\Rightarrow e^{2k} = \frac{1}{9}$ $\Rightarrow k = \frac{1}{2}\ln\frac{1}{9} = -\ln 3$
Write out the particular solution.	$X = 300e^{-t\ln 3}$
b) To find half-life, find when $X = 135$mg.	$135 = 270e^{-t\ln 3} \Rightarrow \frac{1}{2} = e^{-t\ln 3}$ $\Rightarrow -\ln 2 = -t\ln 3$ $\Rightarrow t = \frac{\ln 2}{\ln 3}$
Convert to days.	Days = $7 \times \frac{\ln 2}{\ln 3} = 4.41\ (3s.f)$

Example 5: At $t = 0$s a skydiver jumps out of a plane with an initial downward velocity of $v = 2\text{ms}^{-1}$. The skydiver has a mass of 100kg. The skydiver experiences a drag force of $20v\ N$. Set up and solve a differential equation to find the skydiver's terminal velocity. Assume $g = 10\text{ms}^{-2}$ and acts positively downwards. (Where terminal velocity is the constant final velocity that the skydiver reaches)

Use Newton's second law, $F = ma$, to set up a differential equation.	$100g - 20v = 100\frac{dv}{dt}$
Simplify, classify and rewrite the differential equation in a convenient form. Equation is 1st order linear inhomogeneous so use an integrating factor.	$\frac{dv}{dt} + \frac{v}{5} = g$
Use $I(t) = e^{\int P(t)\,dt}$ to find $I(t)$.	$I(t) = e^{\int \frac{1}{5}dt} = e^{\frac{t}{5}}$
Multiply both sides of the differential equation by $e^{\frac{t}{5}}$.	$e^{\frac{t}{5}}\frac{dv}{dt} + e^{\frac{t}{5}}\frac{v}{5} = ge^{\frac{t}{5}}$
Rewrite LHS using the product rule.	$\frac{d}{dt}\left(e^{\frac{t}{5}}v\right) = ge^{\frac{t}{5}}$
Integrate both sides.	$e^{\frac{t}{5}}v = 5ge^{\frac{t}{5}} + c$
Divide both sides by $e^{\frac{t}{5}}$ to obtain the general solution.	$v = 5g + ce^{-\frac{t}{5}}$
Substitute in initial conditions to fix the arbitrary constant.	$2 = 5g + ce^{-\frac{0}{5}} \Rightarrow c = 2 - 5g = -48$
Substitute $c = -48$ into general solution to get the particular solution.	$v = 5g - 48e^{-\frac{t}{5}}$
Take limit as t → ∞ to find the terminal velocity.	$v = \lim_{t\to\infty}\left(5g - 48e^{-\frac{t}{5}}\right) = 5g = 50ms^{-1}$